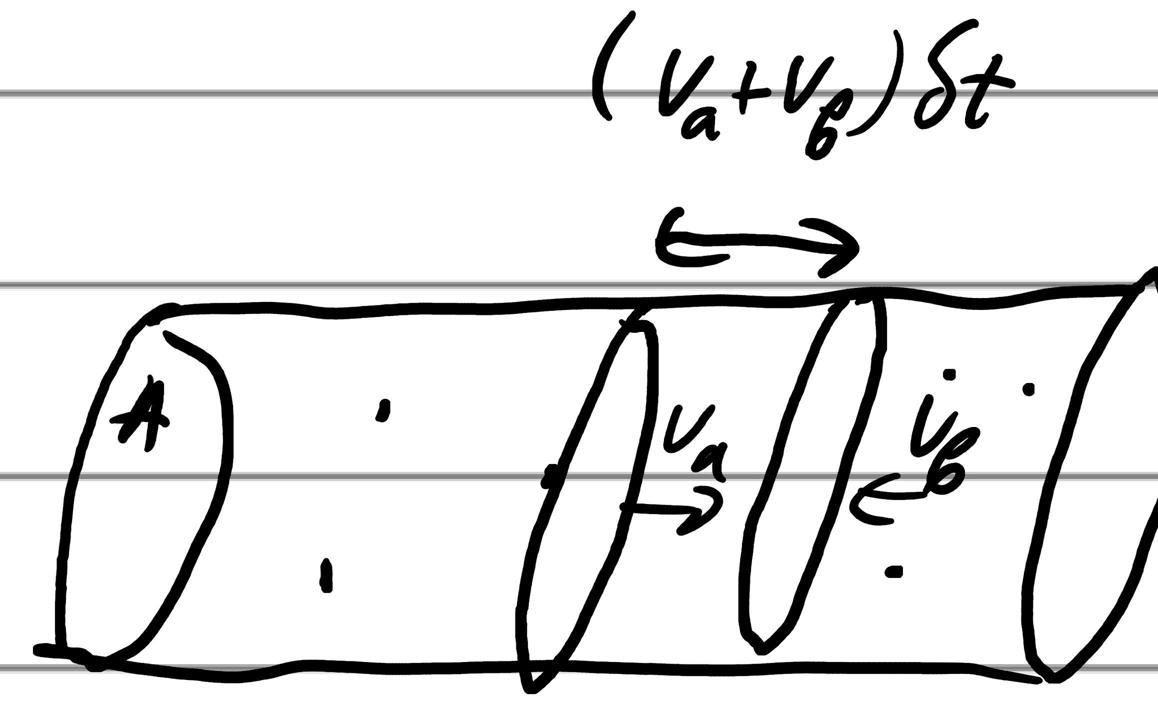
Cross section

Particle a with velocity v_a
Target particle b with v_b



In time δt a traverses region with # of particles b:
$(v_a+v_b)\delta t \cdot n_b \cdot A$

Probability for int is ~ the cross sectional area occupied by $(v_a+v_b)\delta t\, n_b A$ particles

$$\frac{(v_a+v_b)\delta t\, n_b\, A\,\sigma}{A} = v \cdot n_b\, \delta t\, \sigma$$

$\Rightarrow$ rate (1/s) $= v \cdot n_b \cdot \sigma$ (rate per particle a)

Take volume V: $V \cdot v \cdot n_b \cdot \sigma \cdot n_a = (n_b \cdot V)(n_a \cdot V) \cdot \sigma =$
$= N_b\, \phi_a \cdot \sigma$ $\leftarrow$ cross section $=$ Rate

total number of b (N_b), flux of a (ϕ_a)

$$\text{Rate}/\text{Volume} = \overset{v_1+v_2}{v} \cdot n_1 \cdot n_2 \cdot V\sigma$$

For 1 particle per unit volume the rate is $(n_1 = n_2 = 1)$

$$\Gamma_{fi} = (v_1+v_2)\,\sigma \Rightarrow \sigma = \frac{\Gamma_{fi}}{v_1+v_2}$$

$$\Rightarrow \sigma = \frac{1}{4\pi^2} \cdot \frac{1}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4)\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4}$$

$$\underbrace{2E_1 2E_2 (v_1 + v_2)}_{= F}$$

$$F = 4\sqrt{p_1^\mu p_{2\mu} - m_1^2 m_2^2} \leftarrow LI$$

CoM frame:

$$F = 4E_1E_2\left(\frac{p_i^*}{E_1} + \frac{p_i^*}{E_2}\right) = 4p_i^*(E_1 + E_2) = 4p_i^* \sqrt{s}$$

Target particle (particle 2 at rest)

$$F = 4E_1E_2 \frac{|\vec{p}_1|}{E_1} = 4m_2|\vec{p}_1|$$

$\boxed{\text{CoM frame}}$ $\quad \vec{p}_1 + \vec{p}_2 = 0 \quad E_1 + E_2 = \sqrt{s}$

$$\Rightarrow \sigma = \frac{1}{4\pi^2} \cdot \frac{1}{4p^*\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_3 - E_4)\, \delta^3(\vec{p}_3 + \vec{p}_4) \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4}$$

$$p^* = |\vec{p}_i^*|$$

Exactly the same integral as Γ_{fi} $\quad m_i = \sqrt{s}$

$$\Rightarrow \sigma = \frac{1}{16\pi^2} \cdot \frac{1}{|\vec{p}_i^*|\sqrt{s}} \cdot \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|\, d\Omega^* = \frac{1}{64\pi^2 s} \cdot \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*$$

Elastic scattering $|\vec{p}_i^{\,*}| = |\vec{p}_f^{\,*}|$

$$\sigma_{elastic} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 d\Omega^*$$

For LI x-section the previous result is sufficient
total

Not useful for computing $\frac{d\sigma}{d\Omega}$ - in a $\neq$ than CoM frame

$d\Omega^* = d(\cos\theta^*) d\varphi^*$ → angles in CoM frame

$$d\sigma = \frac{1}{64\pi^2 s} \cdot \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |M_{fi}|^2 d\Omega^*$$ Look for LI expression for $d\sigma$

Express $d\Omega^*$ in terms of t

$$t = q^2 = (p_1 - p_3)^2 = m_1^2 + m_2^2 - 2p_1 \cdot p_3$$

In CoM: $p_1^* = (E_1^*, 0, 0, |\vec{p}_1^{\,*}|)$ $p_3^* = (E_3^*, |\vec{p}_3^{\,*}|\sin\theta^*, 0, |\vec{p}_3^{\,*}|\cos\theta^*)$

$\Rightarrow p_1 \cdot p_3 = E_1^* E_3^* - |\vec{p}_1^{\,*}||\vec{p}_3^{\,*}|\cos\theta^* \Rightarrow$

$\Rightarrow t = m_1^2 + m_2^2 - 2E_1^* E_3^* - 2|\vec{p}_1^{\,*}||\vec{p}_3^{\,*}|\cos\theta^*$

$dt = 2|\vec{p}_1^{\,*}||\vec{p}_3^{\,*}|\, d(\cos\theta^*)$

$$d\Omega^* = d(\cos\theta^*) d\phi^* = \frac{dt\, d\phi^*}{2|\vec{p}_1^{\,*}||\vec{p}_3^{\,*}|}$$

$$d\sigma = \frac{1}{64\pi s |\vec{p}_1^{\,*}|^2} |M_{fi}|^2 dt$$ (after integrating over ϕ^*)

$|\vec{p}_1^{\,*}|$ - fixed by E-M conservation

$$|\vec{p}_1^{\,*}|^2 = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right] = (s-m_2^2)^2/4s$$

neglect m_1 (ℓ or ν scattering)

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{16\pi(s-m_2^2)^2}|M_{fi}|^2}$$

HE elastic scattering: $m_1 = m_3 = 0$ $m_2 = m_4 = M$

Lab. frame

$d\Omega = 2\pi d(\cos\theta)$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\cdot\frac{dt}{d\Omega} = \frac{d\sigma}{dt}\cdot\frac{1}{2\pi}\frac{dt}{d(\cos\theta)}$$

$p_1 = (E_1, 0, 0, E_1)$ $p_2 = (M, 0, 0, 0)$, $p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)$

$p_4 = (E_4, \vec{p}_4)$

(2) $t = (p_1 - p_3)^2 = -2p_1p_3 = -2E_1E_3(1-\cos\theta)$

From energy-momentum conservation: $p_1 + p_3 = p_2 + p_4$

(3) $t = (p_2 - p_4)^2 = 2M^2 - 2ME_4 = -2M(M - E_4) = -2M(E_1 - E_3)$

E_1 is consrant (energy of incoming particle)

$\Rightarrow \dfrac{dt}{d(\cos\theta)} = 2M \dfrac{dE_3}{d(\cos\theta)}$

From (2) = (3) we get $E_3 = \dfrac{E_1 M}{M + E_1(1-\cos\theta)}$

$$\frac{dE_3}{d(\cos\theta)} = \frac{E_1^2 M}{(M + E_1(1-\cos\theta))^2} = E_1^2 M \left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)} \cdot \frac{d\sigma}{dt} = \frac{1}{\cancel{2}\pi} \cdot \cancel{2}\cancel{M} \cdot \frac{E_3^2}{\cancel{M}}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \cdot \frac{d\sigma}{dt}$$

$$= \frac{E_3^2}{16\pi^2 (s-M)^2} \cdot |M_{fi}|^2$$; use $s = (p_1 + p_2)^2 = M^2 + 2p_1p_2 =$
$= M^2 + 2E_1 M \Rightarrow (s - M^2) = 2E_1 M$

$\Downarrow$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \left(\frac{E_3}{E_1 M}\right)^2 |M_{fi}|^2 \quad (\text{for } m_1 \to 0)}$$

E_3 as a function of ϑ: $E_3 = \dfrac{E_1 M}{M + E_1(1-\cos\vartheta)}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\left(\frac{1}{M+E_1(1-\cos\vartheta)}\right)^2 |M_{fi}|^2$$

for general $2 \to 2$ scattering

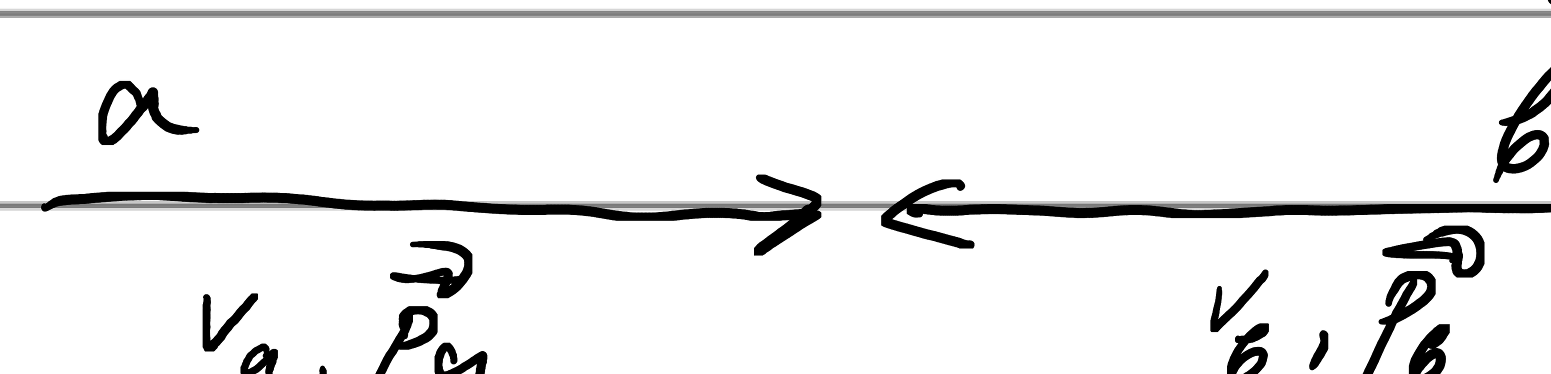
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{|\vec{p}_1| m_1}\,\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\vartheta}\,|M_{fi}|^2$$

Use $\underbrace{(E_1+m_2)}_{=\sqrt{s}} = \underbrace{\sqrt{|\vec{p}_3|^2+m_3^2}}_{E_3} + \sqrt{\underbrace{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\vartheta}_{\vec{p}_4=\vec{p}_1-\vec{p}_3}+m_4^2}$

$\Rightarrow |\vec{p}_3|$ is a function of ϑ

Lorentz-invariant flux: $F = 4E_aE_b(v_a+v_b) = 4E_aE_b\left(\dfrac{|\vec{p}_a|}{E_a}+\dfrac{|\vec{p}_b|}{E_b}\right)$

$= 4(|\vec{p}_a|E_b + E_a|\vec{p}_b|)$



$$p_a p_b = p_a^\mu p_{b\mu} = E_aE_b - \vec{p}_a\cdot\vec{p}_b = E_aE_b + |\vec{p}_a||\vec{p}_b|$$

$$F^2/16 - (p_a^\mu p_{b\mu})^2 = -(E_aE_b + |\vec{p}_a|.|\vec{p}_b|) + (E_a|\vec{p}_b| + E_b|\vec{p}_a|)^2 =$$

$$= |\vec{p}_a|^2 \left(E_b^2 - |\vec{p}_b|^2 \right) + E_a^2 \left(|\vec{p}_b|^2 - E_b^2 \right) = |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 = -m_a^2 m_b^2$$

$$\Rightarrow F = 4\sqrt{(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2}$$